Interactions among Peakons, Dromions, and Compactons in (2+1)-Dimensional System

Cheng-lin Bai, Hai-quan Hu, Wen-jun Wang, and Hong Zhao

Physics Science and Information Engineering School, Liaocheng University, Shandong 252059, China

Reprint requests to Dr. C.B.; E-mail: lcced_bcl@lctu.edu.cn or lcced_bcl@hotmail.com (Ch.Bai)

Z. Naturforsch. **59a,** 729 – 742 (2004); received July 20, 2004

Starting from a quite universal formula, which is valid for some quite universal (2+1)-dimensional physical models, the interactions among different types of solitary waves like peakons, dromions, and compactons are investigated both analytically and graphically. Some novel and interesting features are revealed. The results show that the interactions for peakon-dromion, compacton-dromion, and peakon-compacton may be completely inelastic, not-completely elastic or completely elastic. – PACS numbers: 05.45.Yv, 02.30.Jr.

Key words: Interaction; Peakon; Dromion; Compacton; (2+1)-dimensional System.

1. Introduction

In the study of nonlinear science, soliton theory plays a very important role and has been applied in almost all natural sciences, especially in all physics branches, such as condensed matter physics, field theory, fluid dynamics, plasma physics, optics, etc. [1]. (1+1)-dimensional solitons and solitary wave solutions have been studied quite well both theoretically and experimentally, for example, dromions (exponentially localized in all directions), compactons (this type of solutions describes the typical (1+1)-dimensional soliton solutions with finite wavelength, when the nonlinear dispersion effects are included in the models) and peakons (a special type of weak solutions of the (1+1)-dimensional Camassa-Holm (CH) equation, which are discontinuous at their crest). From the symmetry study of the (2+1)-dimensional integrable models we know that in higher dimensions there exist richer symmetry structures than in lower dimensions [2]. This suggests that the solitary waves structures and the interactions between solitary waves of the (2+1)-dimensional nonlinear models may shows quite rich phenomena that have not yet been revealed. Almost all previous studies of interactions among solitary waves, especially in higher dimensions, are restricted to the same type of localized structures. Thus the peakon-peakon and dromion-dromion interactions have been investigated, and the results indicate that the former are not completely elastic while the latter are completely elastic, and that for some types of compactons the interactions among them are not completely elastic while for some others the interactions are completely elastic, but the interactions among the different types of solitary waves, like peakons-dromions, compactons-dromions, and peakons-compactons, have not yet been often studied.

Motivated by these reasons, we have taken the (2+1)-dimensional soliton equation

$$iq_t + q_{xx} + qR = 0, \tag{1}$$

$$R_t + R_y + (qq^*)_x = 0, \tag{2}$$

which is derived from the Kadomtsev-Petviashvili (KP) equation by using an asymptotically exact reduction method based on Fourier expansion and spatiotemporal rescaling [3], as a concrete example. The integrability of the equations has been discussed by Porsezian [4]. Starting from a special Bäcklund transformation, we convert (1) and (2) into simple variable separated equations, and then obtain a rather general variable separated solution. This solution turns and to be a quite "universal" formula, and is valid for suitable fields or potentials of various (2+1)-dimensional physically interesting integrable models, including the Davey-Stewartson (DS) equation, the dispersive long wave length equation (DLWE) [5], the Broer-Kaup (BK) system [6, 7], the higher-order Broer-Kaup (HBK) system [8], the Nizhnik-Novikov-

Vesselov (NNV) system, the ANNV (asymmetric NNV) equation, and so on [9].

The paper is organized as follows. In Sect. 2, we apply a variable separation approach to solve the (2+1)-dimensional soliton equation and obtain its exact excitation. In Sect. 3, we analyze interaction properties among different types of coherent soliton structures. A brief discussion and summary is given in the last section.

2. Variable Separated Solutions for a (2+1)-Dimensional Soliton Equation

To find the interesting localized structures in the solutions of the system (1) and (2), first we make the following variable transformation

$$\eta = y+t, \quad \tau = t-y, \quad x = x. \tag{3}$$

Eqs. (1) and (2) then read as

$$i(q_\eta + q_\tau) + q_{xx} + qR = 0, \tag{4}$$

$$R_\eta + \frac{1}{2}(qq^*)_x = 0. \tag{5}$$

Then we take the following Bäcklund transformation

$$q = \frac{g}{f} + q_0, \quad R = 2(\ln f)_{xx} + R_{0x}, \tag{6}$$

where f is real, g is complex, and (q_0, R_0) are arbitrary seed solutions. Under the transformation (6), (4) and (5) are transformed to their bilinear form

$$\begin{aligned}&(D_x^2 - iD_\eta - iD_\tau) f\cdot g + q_0 D_x^2 f\cdot f + fgR_{0x}\\&\quad + f^2 q_0 R_{0x} + f^2 q_{0xx} + i f^2 (q_{0\eta} + q_{0\tau}) = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}&2D_x D_\eta f\cdot f + gg^* + fg^* q_0 + fgq_0^* + f^2 q_0 q_0^*\\&\quad + 2f^2 R_{0\eta} = 0,\end{aligned} \tag{8}$$

where D_x, D_η are the usual bilinear operator, defined as

$$D_x^m D_\eta^n f\cdot g = \lim_{x'=x,\eta'=\eta,\tau'=\tau} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \eta'}\right)^n \cdot f(x,\eta,\tau)\cdot g(x',\eta',\tau').$$

To discuss further, we fix the seed solution (q_0, R_0) as

$$q_0 = 0, R_0 = u_0(x,\tau). \tag{9}$$

Eqs. (7) and (8) can be simplified to

$$(D_x^2 - iD_\eta - iD_\tau) f\cdot g + fgR_{0x} = 0, \tag{10}$$

$$2D_x D_\eta f\cdot f + gg^* = 0. \tag{11}$$

To solve the bilinear (10) and (11) with (9), we make the ansatz

$$\begin{aligned}&f = a_1 u(x,\tau) + a_2\varphi(\eta,\tau) + a_3 u(x,\tau)\varphi(\eta,\tau),\\&g = u_1(x,\tau)\varphi_1(\eta,\tau)\exp\left(ir(x,\tau) + is(\eta,\tau)\right),\end{aligned} \tag{12}$$

where a_1, a_2, and a_3 are arbitrary constants and u, φ, u_1, φ_1, r, s are all real functions of the indicated variables. Substituting (12) into (10) and (11) and separating the real and imaginary parts of the resulting equation, we have

$$u_1^2\varphi_1^2 - 4a_1 a_2 u_x \varphi_\eta = 0, \tag{13}$$

$$\begin{aligned}&(a_1 + a_3\varphi)(2u_x u_{1x} - u_1 u_{xx}) + (a_1 u + a_2\varphi + a_3 u\varphi)\\&\cdot(u_1 r_\tau + u_1 s_\tau - u_1 u_{0x} + u_1 r_x^2 + u_1 s_\eta - u_{1xx}) = 0,\end{aligned} \tag{14}$$

$$\begin{aligned}&(a_1 + a_3\varphi)(u_\tau + 2u_x r_x)u_1\varphi_1\\&\quad + (a_2 + a_3 u)(\varphi_\eta + \varphi_\tau)u_1\varphi_1\\&\quad - (a_1 u + a_2\varphi + a_3 u\varphi)(\varphi_1 u_{1\tau} + u_1\varphi_{1\tau} + u_1\varphi_{1\eta}\\&\quad + 2\varphi_1 u_{1x} r_x + u_1\varphi_1 r_{xx}) = 0.\end{aligned} \tag{15}$$

Because the functions u_0, u, u_1, and r are only functions of $\{x,\tau\}$, and the functions φ, φ_1, and s are only functions of $\{\eta,\tau\}$, (13)–(15) can be solved by the following variable separated equations

$$u_1 = 2\delta_1\sqrt{a_1 a_2 c_0^{-1} u_x}, \tag{16}$$

$$\varphi_1 = \delta_2\sqrt{c_0\varphi_\eta}, \quad \left(\delta_1^2 = \delta_2^2 = 1\right), \tag{17}$$

$$s_\eta + s_\tau = 0, \tag{18}$$

$$4r_\tau u_x^2 + 4u_x^2 r_x^2 + u_{xx}^2 - 4u_x^2 u_{0x} - 2u_x u_{xx} = 0, \tag{19}$$

$$\begin{aligned}&u_\tau + 2u_x r_x =\\&\quad c_1\left(a_2 + a_3 u\right)^2 + c_2\left(a_2 + a_3 u\right) + a_1 a_2 c_3,\end{aligned} \tag{20}$$

$$\begin{aligned}&\varphi_\eta + \varphi_\tau =\\&\quad -c_3\left(a_1 + a_3\varphi\right)^2 - c_2\left(a_1 + a_3\varphi\right) - a_1 a_2 c_1.\end{aligned} \tag{21}$$

In (16)–(21), c_0, c_1, c_2, and c_3 are all functions of τ. In (16) and (17), the conditions of u and φ to be real require

$$a_1 a_2 c_0^{-1} u_x \geq 0, \tag{22}$$

$$c_0\varphi_\eta \geq 0. \tag{23}$$

Actually, due to the arbitrariness of the function u_0, the localized solutions of (1) and (2) possess very rich structures. According to (18) and (20) it is obvious that $s \equiv s(\eta-\tau)$ is an arbitrary function of $(\eta-\tau)$, and the function r can be expressed in terms of u. Moreover, the arbitrary function u_0 is related to u according to (19). In fact, we can treat the problem alternatively by considering u to be an arbitrary function of x and τ and fixing the function u_0 by (19). The result reads

$$u_{0x} = (4u_x^2)^{-1}(4r_\tau u_x^2 + 4u_x^2 r_x^2 + u_{xx}^2 - 2u_x u_{xx}). \tag{24}$$

Now substituting (12) with (16) – (21) into (6), we get a general solution of the system (1) and (2):

$$q = \frac{2\delta_1\delta_2\sqrt{a_1a_2u_x\varphi_\eta}\exp(ir+is)}{a_1u+a_2\varphi+a_3u\varphi}, \tag{25}$$

$$R = u_{0x} + 2\Big(\frac{a_1u_{xx}+a_3\varphi u_{xx}}{a_1u+a_2\varphi+a_3u\varphi} - \frac{(a_1u_x+a_3\varphi u_x)^2}{(a_1u+a_2\varphi+a_3u\varphi)^2}\Big) \tag{26}$$

with two arbitrary functions u and φ under the conditions (22) and (23), and u_0 is determined by (24). Especially, the modulus squared of the field q reads

$$Q = |q|^2 = \frac{4a_1a_2u_x\varphi_\eta}{(a_1u+a_2\varphi+a_3u\varphi)^2} = \frac{a_1a_2U_xV_\eta}{\big(A_1\cosh\frac{1}{2}(U+V+C_1)+A_2\cosh\frac{1}{2}(U-V+C_2)\big)^2}, \tag{27}$$

where

$$\begin{aligned} &u = b_1 + e^U, \varphi = b_2 + e^V, \\ &A_1 = \sqrt{a_3(a_1b_1+a_2b_2+a_3b_1b_2)}, \\ &A_2 = \sqrt{(a_1+a_3b_2)(a_2+a_3b_1)}, \\ &C_1 = \ln\frac{a_3}{a_1b_1+a_2b_2+a_3b_1b_2}, \\ &C_2 = \ln\frac{a_1+a_3b_2}{a_2+a_3b_1}, \end{aligned} \tag{28}$$

and b_1 and b_2 being arbitrary constants. U and V are also arbitrary functions of (x,τ) and $(\eta-\tau,\tau)$, respectively, under the conditions $a_1a_2U_xV_\eta \geq 0$.

Because of the arbitrariness of the functions of $u(x,\tau)$ and $\varphi(\eta-\tau,\tau)$, (27) reveals quite a variety of soliton structures. Actually, from (27) it is easy to show that the arbitrary functions u and φ with the boundary conditions

$$\begin{aligned} &u\,|_{x\to-\infty} \to B_1, \quad u\,|_{x\to+\infty} \to B_2, \\ &\varphi\,|_{\eta\to-\infty} \to B_3, \quad \varphi\,|_{\eta\to+\infty} \to B_4, \end{aligned} \tag{29}$$

where B_1, B_2, B_3, and B_4 are arbitrary constants which may be infinite, are coherent soliton solutions localized in all directions.

3. Interactions among Peakons, Dromions, and Compactons in a (2+1)-Dimensional System

It is interesting that the expression (27) is valid for many (2+1)-dimensional models like the DS equation, NNV system, ANNV equation, BK equation, etc. [5 – 9]. Because of the arbitrariness of the functions u and φ, included in (27), the quantity Q possesses quite rich structures. For instance, if we select the functions u and φ appropriately, we can obtain many kinds of localized solutions, like multi-solitoff solutions, multi-dromion and dromion lattice solutions, multiple ring soliton solutions, peakons, compactons and so on. The properties of peakon-peakon, dromion-dromion, and compacton-compacton interactions were also discussed in [6 – 9]. Now we pay our attention to interactions of different types of solitary waves, i. e., the interactions among peakons, dromions, and compactons.

3.1. *Asymptotic Behavior of the Localized Excitations Produced from (27) [10, 11]*

In general, if the functions u and φ are selected as localized solitonic excitations with

$$u\Big|_{\tau\to\mp\infty} = \sum_{i=1}^{M} u_i^\mp, \quad u_i^\mp \equiv u_i(x-c_i\tau+\delta_i^\mp), \tag{30}$$

$$\varphi\Big|_{\tau\to\mp\infty} = \sum_{j=1}^{N} \varphi_j^\mp, \quad \varphi_j^\mp \equiv \varphi_j(\eta-C_j\tau+\Delta_j^\mp), \tag{31}$$

where $\{u_i,\varphi_j\}\,\forall i$ and j are localized functions, then the physical quantity Q expressed by (27) delivers $M\times N$ (2+1)-dimensional localized excitations with the asymptotic behaviour

$$Q|_{\tau\to\mp\infty} \to \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{4a_1a_2u_{ix}^{\mp}\varphi_{j\eta}^{\mp}}{\left(a_1\left(u_i^{\mp}+U_i^{\mp}\right)+a_2\left(\varphi_j^{\mp}+\Phi_j^{\mp}\right)+a_3\left(u_i^{\mp}+U_i^{\mp}\right)\left(\varphi_j^{\mp}+\Phi_j^{\mp}\right)\right)^2}$$
$$\equiv \sum_{i=1}^{M}\sum_{j=1}^{N} Q_{ij}^{\mp}\left(x-c_i\tau+\delta_i^{\mp}, \eta-C_i\tau+\Delta_i^{\mp}\right) \equiv \sum_{i=1}^{M}\sum_{j=1}^{N} Q_{ij}^{\mp}, \tag{32}$$

where

$$U_i^{\mp} = \sum_{j<i} u_j(\mp\infty) + \sum_{j>i} u_j(\pm\infty), \tag{33}$$

$$\Phi_i^{\mp} = \sum_{j<i} \varphi_j(\mp\infty) + \sum_{j>i} \varphi_j(\pm\infty), \tag{34}$$

and we have assumed without loss of generality, $C_i > C_j$ and $c_i > c_j$ if $i > j$.

It can be deduced from (32) that the ij th localized excitation Q_{ij} preserves its shape during the interaction iff

$$U_i^+ = U_i^-, \tag{35}$$

$$\Phi_j^+ = \Phi_j^-. \tag{36}$$

Meanwhile, the phase shift of the ij th localized excitation Q_{ij} reads

$$\delta_i^+ - \delta_i^- \tag{37}$$

in the x direction and

$$\Delta_j^+ - \Delta_j^- \tag{38}$$

in the η direction.

The above discussions demonstrate that localized solitonic excitations for the universal quantity Q can be constructed without difficulties via the (1+1)-dimensional localized excitations with the properties (30), (31), (35), and (36). As a matter of fact, any localized solutions (or their derivatives) with completely elastic (or not completely elastic or completely inelastic) interaction behavior of any known (1+1)-dimensional integrable models can be utilized to construct (2+1)-dimensional localized solitonic solutions with completely elastic ($U_i^+ = U_i^-$, $\Phi_j^+ = \Phi_j^-$ for all i,j), or not completely elastic or completely inelastic ($U_i^+ \neq U_i^-$, $\Phi_j^+ \neq \Phi_j^-$ at least for one of i,j) interaction properties. However, to the best of our knowledge, the interactions among different types of solitary waves like peakons-dromions, compactons-dromions, and peakons-compactons were not reported in the literature. In order to see the interaction behavior among different types of solitary waves more directly, we investigate some special examples by fixing the arbitrary functions u and φ in (27).

3.2. Completely Inelastic Interactions

We first discuss the peakon-dromion interaction. As is known, if selecting u and φ to be some piecewise smooth functions, then one can derive some multi-peakon excitation [6–9]. For instance, when u and φ are taken to have the simple forms

$$\begin{aligned} u &= a_0 + \sum_{i=1}^{M} d_i \exp(m_i x - \beta_i\tau + x_{0i}), \\ &\quad \text{if } m_i x - \beta_i\tau + x_{0i} \le 0, \\ u &= a_0 + \sum_{i=1}^{M} \left(-d_i \exp(-m_i x + \beta_i\tau - x_{0i}) + 2d_i\right), \\ &\quad \text{if } m_i x - \beta_i\tau + x_{0i} > 0, \end{aligned} \tag{39}$$

$$\begin{aligned} \varphi &= \sum_{j=1}^{N} e_j \exp(n_j\eta + w_j\tau + \eta_{0j}), \\ &\quad \text{if } n_j\eta + w_j\tau + \eta_{0j} \le 0, \\ \varphi &= \sum_{j=1}^{N} \left(-e_j \exp(-n_j\eta - w_j\tau - \eta_{0j}) + 2e_j\right), \\ &\quad \text{if } n_j\eta + w_j\tau + \eta_{0j} > 0, \end{aligned} \tag{40}$$

where d_i, m_i, β_i, e_j, n_j, x_{0i} and η_{0j} are all arbitrary constants, the solution (27) with (39) and (40) becomes a multi-peakon solution.

The dromion solution, that is localized exponentially in all directions, can be derived by multiple straight-line and curved-line ghost solitons. As simple choices for the functions u and φ one can take

$$u = 1 + \sum_{i=1}^{M} \exp\left[K_i(x+\beta\tau) + x_{1i}\right], \tag{41}$$

$$\varphi = 1 + \sum_{j=1}^{N} \exp\left[k_j(\eta+w\tau) + \eta_{1j}\right], \tag{42}$$

where K_i, k_j, x_{1i}, and η_{1j} are arbitrary constants and M, N are positive integers.

Now the key problem is how to construct the peakon and dromion structures for the physical quantity Q simultaneously. Since there may exist peakons and dromions at the same time in the real natural phenomena, we can take u and φ to have the simple forms

$$u = 1+\sum_{i=1}^{M}\exp\left[K_i\left(x+\beta\tau\right)+x_{1i}\right] + \begin{cases} a_0+\sum_{i=1}^{P} d_i\exp(m_i x-\beta_i\tau+x_{0i}), \\ \quad \text{if } m_i x-\beta_i\tau+x_{0i}\le 0, \\ a_0+\sum_{i=1}^{P}(-d_i\exp(-m_i x+\beta_i\tau-x_{0i}+2d_i), \\ \quad \text{if } m_i x-\beta_i\tau+x_{0i}> 0, \end{cases} \tag{43}$$

$$\varphi = 1+\sum_{j=1}^{N}\exp\left[k_j\left(\eta+w\tau\right)+\eta_{1j}\right], \tag{44}$$

where M, N, and P are positive integers. Then we may construct peakon and dromion structures simultaneously for the physical quantity Q. These selections (43) and (44) (related to a completely inelastic interaction) are new and different from the selections of (132) and (133) of [10].

For simplicity, we take

$$\begin{aligned} &M=N=P=1,\ K_1=k_1=1,\ \beta=1, \\ &w=x_{11}=\eta_{11}=0,\ a_0=0,\ a_1=a_2=a_3=1, \\ &d_1=m_1=1,\ \beta_1=1/2,\ x_{01}=5, \end{aligned} \tag{45}$$

and obtain a combined localized coherent structure depicted in Figure 1. From Fig. 1 we can see that the peakon-dromion interaction is completely inelastic, which is very similar to the completely inelastic collisions between two classical particles.

Along the same line of argument, and performing a similar analysis, when u and φ are taken as the simple forms

$$\begin{aligned} &u=a_0,\ \text{if } x+\beta_i\tau\le x_{0i}-\frac{\pi}{2k_i}, \\ &u=a_0+\sum_{i=1}^{M}\left(b_i\sin\left(k_i(x+\beta_i\tau-x_{0i})\right)+b_i\right), \\ &\quad \text{if } x_{0i}-\frac{\pi}{2k_i}<x+\beta_i\tau\le x_{0i}+\frac{\pi}{2k_i}, \\ &u=a_0+\sum_{i=1}^{M}2b_i,\ \text{if } x+\beta_i\tau> x_{0i}+\frac{\pi}{2k_i}, \end{aligned} \tag{46}$$

$$\begin{aligned} &\varphi=0,\ \text{if } \eta+w_j\tau\le\eta_{0j}-\frac{\pi}{2l_j}, \\ &\varphi=\sum_{j=1}^{N}\left(c_j\sin\left(l_j(\eta+w_j\tau-\eta_{0j})\right)+c_j\right), \\ &\quad \text{if } \eta_{0j}-\frac{\pi}{2l_j}<\eta+w_j\tau\le\eta_{0j}+\frac{\pi}{2l_j}, \\ &\varphi=\sum_{j=1}^{N}2c_j,\ \text{if } \eta+w_j\tau>\eta_{0j}+\frac{\pi}{2l_j}, \end{aligned} \tag{47}$$

where b_i, k_i, β_i, c_j, l_j, w_j, x_{0i} and η_{0j} all are arbitrary, then the solution (27) with (46) and (47) becomes a multi-compacton solution. Combining (39) and (40), and taking u and φ to have the forms

$$u = 1+\sum_{i=1}^{M}\exp\left[K_i\left(x+\beta\tau\right)+x_{1i}\right] + \begin{cases} a_0,\ \text{if } x+\beta_i\tau\le x_{0i}-\frac{\pi}{2k_i} \\ a_0+\sum_{i=1}^{P}\left(b_i\sin\left(k_i(x+\beta_i\tau-x_{0i})\right)+b_i\right), \\ \quad \text{if } x_{0i}-\frac{\pi}{2k_i}<x+\beta_i\tau\le x_{0i}+\frac{\pi}{2k_i}, \\ a_0+\sum_{i=1}^{P}2b_i,\ \text{if } x+\beta_i\tau> x_{0i}+\frac{\pi}{2k_i} \end{cases} \tag{48}$$

$$\varphi = \begin{cases} \sum_{j=1}^{N} e_j\exp(n_j\eta+w_j\tau+\eta_{0j}), \\ \text{if } n_j\eta+w_j\tau+\eta_{0j}\le 0 \\ \sum_{j=1}^{N}\left(-e_j\exp(-n_j\eta-w_j\tau-\eta_{0j})+2e_j\right), \\ \text{if } n_j\eta+w_j\tau+\eta_{0j}>0, \end{cases} \tag{49}$$

where M,N, and P are positive integers, then we may construct peakon and compacton structures simultaneously for the physical quantity Q. For convenience, we fix the related parameters as

$$\begin{aligned} &M=N=P=1,\ K_1=1,\ \beta=2,\ x_{11}=0, \\ &a_0=15,\ a_1=a_2=a_3=1, b_1=2,\ k_1=1, \\ &\beta_1=-1,\ x_{01}=0,\ e_1=n_1=1,\ w_1=0,\ \eta_{01}=5, \end{aligned} \tag{50}$$

and find that the peakon-compacton interaction is also completely inelastic. The corresponding plot is depicted in Figure 2.

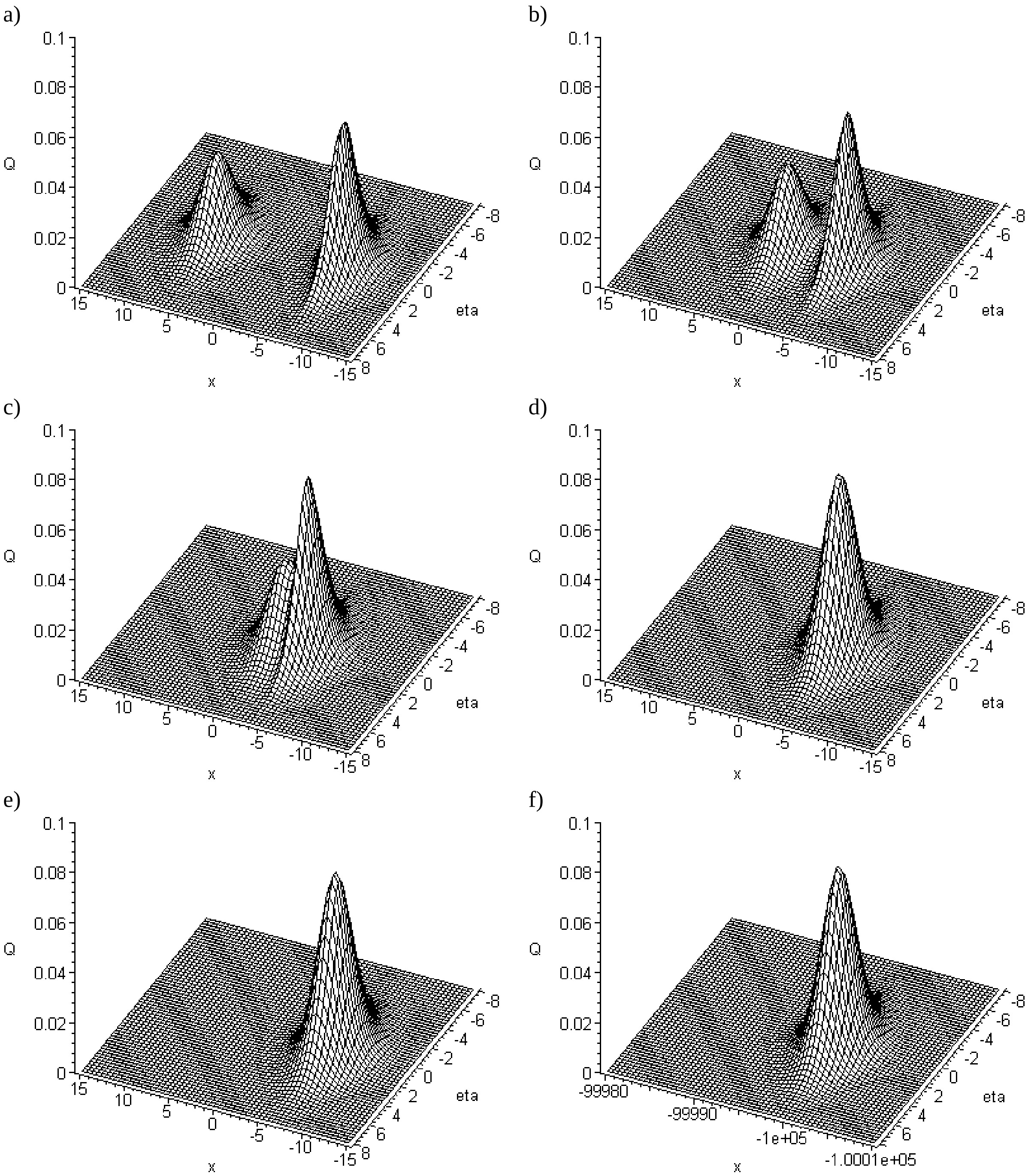


Fig. 1. The evolution of peakon-dromion interaction as seen in the physical quantity Q with the conditions (43), (44) and (45) at the times (a) $\tau = -5$, (b) $\tau = 0$, (c) $\tau = 2.5$, (d) $\tau = 5$, (e) $\tau = 7.5$, and (f) $\tau = 10^5$, respectively.

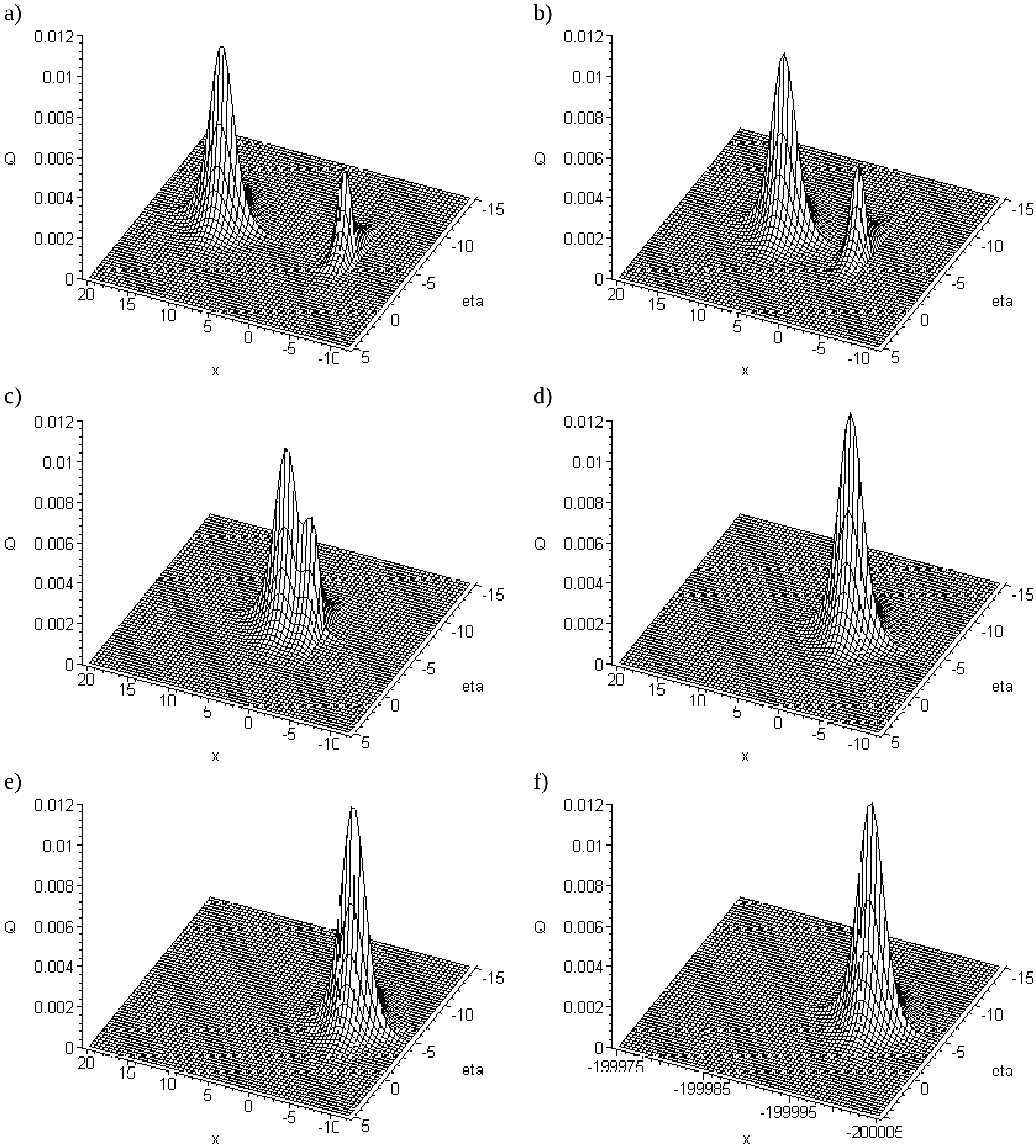


Fig. 2. The temporal evolution of peakon-compacton interaction for the physical quantity Q with the conditions (48), (49) and (50) at the times (a) $\tau = -4$, (b) $\tau = -2$, (c) $\tau = 0$, (d) $\tau = 2$, (e) $\tau = 4$, and (f) $\tau = 10^5$, respectively.

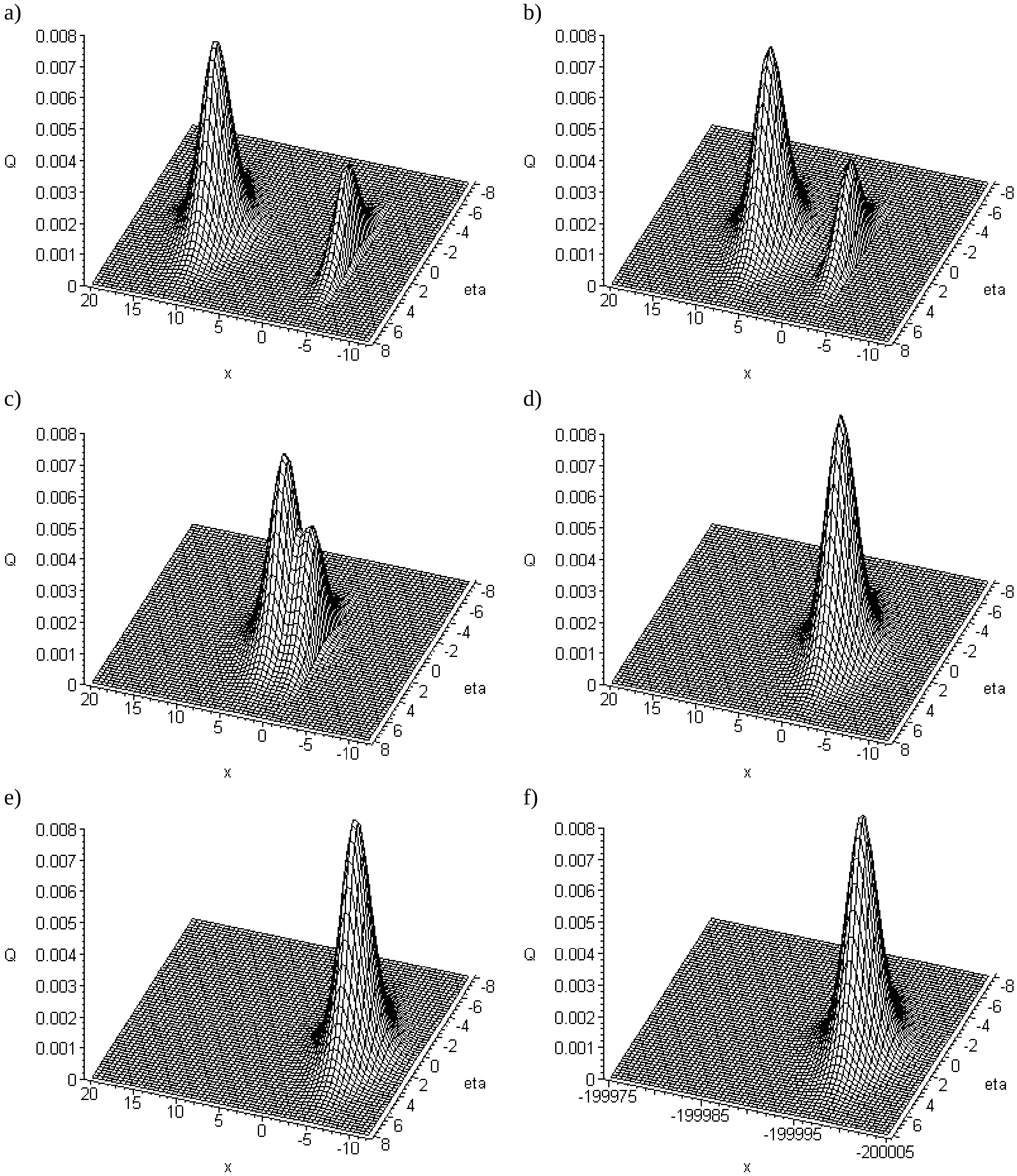


Fig. 3. The time evolution of a dromion-compacton interaction as seen in the physical quantity Q with the conditions (51), (52) and (53) at the times (a) $\tau = -4$, (b) $\tau = -2$, (c) $\tau = 0$, (d) $\tau = 2$, (e) $\tau = 4$, and (f) $\tau = 10^5$, respectively.

According to the above ideas, if we take u and φ to have the forms

$$u = 1 + \sum_{i=1}^{M} \exp\left[K_i\left(x+\beta\tau\right)+x_{1i}\right] + \begin{cases} a_0, \text{ if } x+\beta_i\tau \le x_{0i} - \dfrac{\pi}{2k_i} \\ a_0 + \sum_{i=1}^{P}\left(b_i \sin\left(k_i(x+\beta_i\tau - x_{0i})\right)+b_i\right), \\ \text{if } x_{0i} - \dfrac{\pi}{2k_i} < x+\beta_i\tau \le x_{0i} + \dfrac{\pi}{2k_i}, \\ a_0 + \sum_{i=1}^{P} 2b_i, \text{ if } x+\beta_i\tau > x_{0i} + \dfrac{\pi}{2k_i} \end{cases} \quad (51)$$

$$\varphi = 1 + \sum_{j=1}^{N} \exp\left[k_j\left(\eta + w\tau\right)+\eta_{1j}\right], \quad (52)$$

where M, N, and P are positive integers, then we may construct compacton and dromion structures simultaneously for the physical quantity Q. For convenience we take

$$\begin{aligned} &M = N = P = 1,\ K_1 = k_1 = 1,\ \beta = 2, \\ &w = \eta_{11} = 0, a_0 = 15,\ a_1 = a_2 = a_3 = 1, \\ &b_1 = 2,\ \beta_1 = -1,\ x_{11} = 5,\ x_{01} = 0. \end{aligned} \quad (53)$$

Then we derive a combined localized coherent structure depicted in Figure 3.

3.3. Not-completely Elastic Interactions

It is interesting to mention that though the above choices lead to completely inelastic interaction behaviors for the (2+1)-dimensional solutions, one can also derive some combined localized coherent structures with not-completely elastic interaction behaviors by selecting u and φ appropriately. One of simple choices of the combined localized coherent structures of a dromion and a peakon with not-completely elastic interaction behavior is

$$u = a_0 + \sum_{i=1}^{M} B_i \tanh\left[K_i\left(x+\beta\tau\right)+x_{0i}\right] + \begin{cases} \sum_{i=1}^{P} d_i \exp(m_i x - \beta_i\tau + x_{0i}), \\ \text{if } m_i x - \beta_i \tau + x_{0i} \le 0 \\ \sum_{i=1}^{P}\left(-d_i \exp(-m_i x + \beta_i\tau - x_{0i}) + 2d_i\right), \\ \text{if } m_i x - \beta_i\tau + x_{0i} > 0, \end{cases} \quad (54)$$

$$\varphi = \sum_{j=1}^{N} C_j \exp\left[L_j\left(\eta+\lambda\tau\right)+\eta_{0j}\right] + \begin{cases} \sum_{j=1}^{Q} e_j \exp(n_j\eta + w_j\tau + \eta_{0j}), \\ \text{if } n_j\eta + w_j\tau + \eta_{0j} \le 0 \\ \sum_{j=1}^{Q}\left(-e_j \exp(-n_j\eta - w_j\tau - \eta_{0j}) + 2e_j\right), \\ \text{if } n_j\eta + w_j\tau + \eta_{0j} > 0, \end{cases} \quad (55)$$

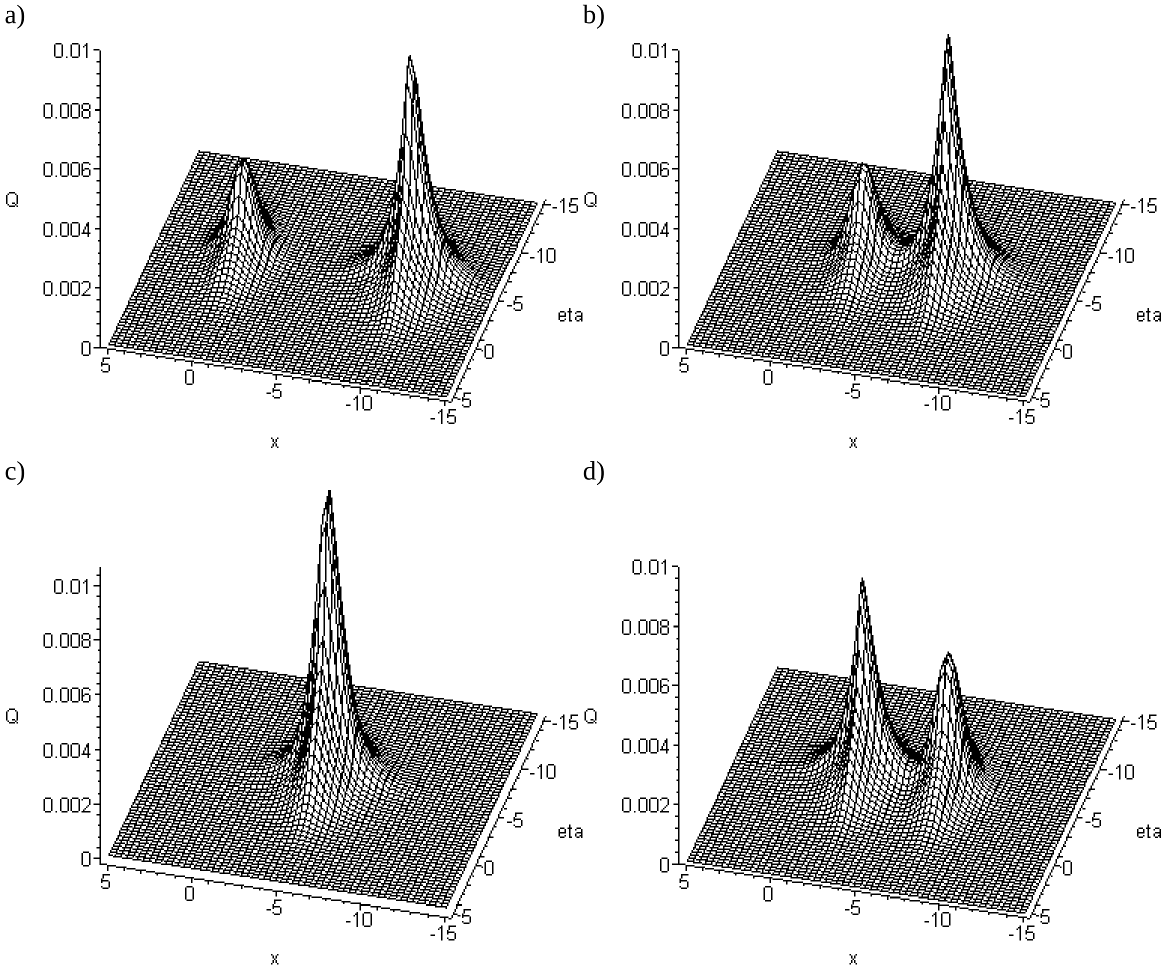
where a_0, B_i, K_i, β, C_j, L_j, λ, d_i, m_i, β_i, e_j, n_j, w_j, x_{0i}, and η_{0j} are all arbitrary constants, and M, N, P, Q are positive integers. The physical quantity Q (27) with (54) and (55) becomes a combined multi-dromion and multi-peakon solution with not-completely elastic interaction behavior. For convenience, we can fix the related parameters as

$$\begin{aligned} &a_0 = 10,\ B_1 = 1/2,\ K_1 = \beta = 1/2, \\ &C_1 = L_1 = 1,\ \lambda = 1, d_1 = m_1 = 1,\ \beta_1 = 1/2, \\ &e_1 = n_1 = 1,\ w_1 = 0,\ x_{01} = 5,\ \eta_{01} = 5, \end{aligned} \quad (56)$$

and obtain a combined localized coherent structure with not-completely elastic interaction behavior as displayed in Figure 4.

Another example is provided by combined dromion and compacton soliton solutions in the (2+1)-dimensional soliton system. The corresponding ansatz is

$$u = a_0 + \sum_{i=1}^{M} B_i \tanh\left[K_i\left(x+\beta\tau\right)+x_{0i}\right] + \sum_{i=1}^{P} \begin{cases} 0, \text{ if } x+\beta_i\tau \le x_{0i} - \dfrac{\pi}{2k_i} \\ b_i \cos^{a_i+1}\left(k_i(x+\beta_i\tau - x_{0i})\right), \\ \text{if } x_{0i} - \dfrac{\pi}{2k_i} < x+\beta_i\tau \le x_{0i} + \dfrac{\pi}{2k_i}, \\ 0, \text{ if } x+\beta_i\tau > x_{0i} + \dfrac{\pi}{2k_i} \end{cases} \quad (57)$$



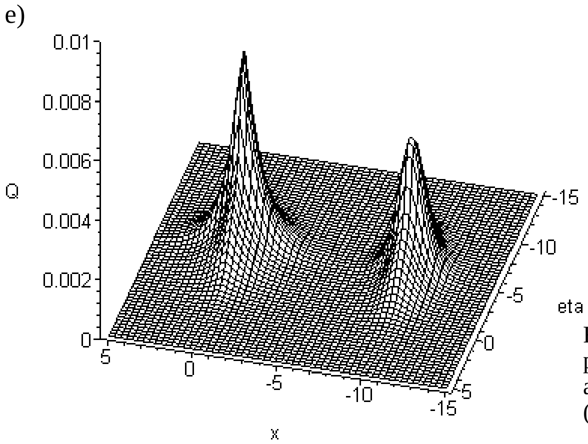
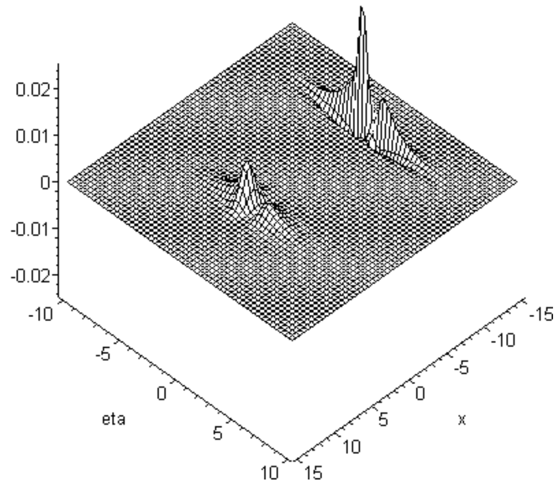
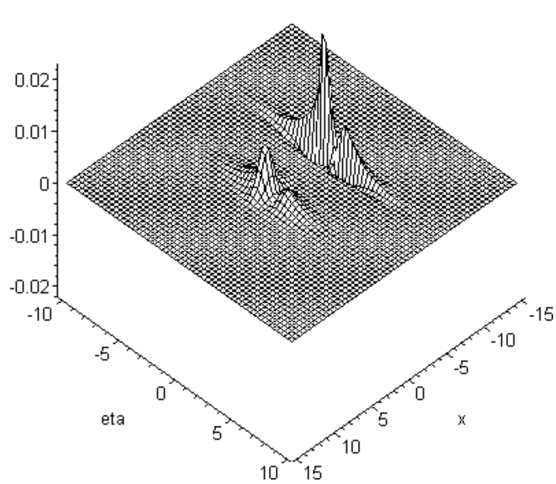


Fig. 4. The evolution of a peakon-dromion interaction in the physical quantity Q with the conditions (54), (55) and (56) at times (a) $\tau = -10$, (b) $\tau = -5$, (c) $\tau = 0$, (d) $\tau = 5$, and (e) $\tau = 10$, respectively.

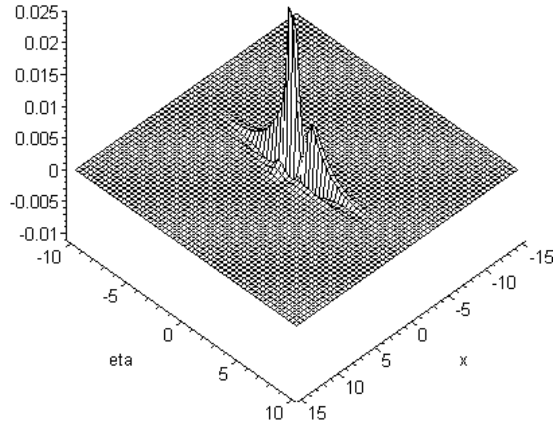
a)



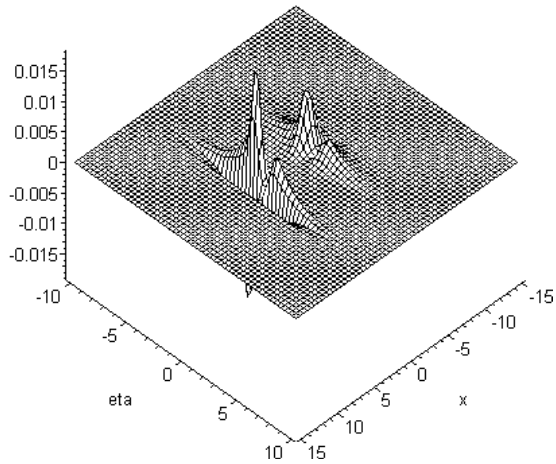
b)



c)



d)



e)

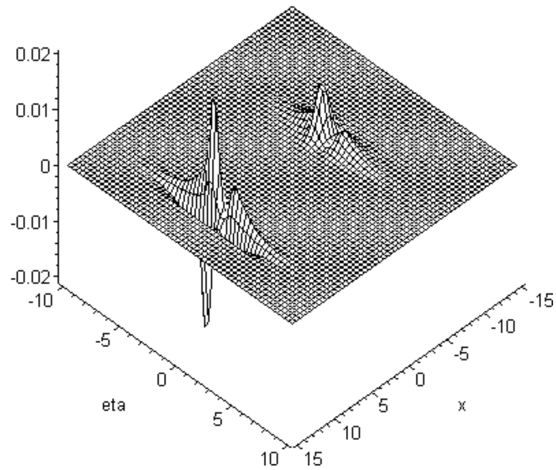


Fig. 5. The evolution of a compacton-dromion interaction for the physical quantity Q with the conditions (57), (58) and (59) at the the times (a) $\tau = -10$, (b) $\tau = -5$, (c) $\tau = 0$, (d) $\tau = 5$, and (e) $\tau = 10$, respectively.

$$\varphi = \sum_{j=1}^{N} C_j \exp\left[L_j\left(\eta + \lambda\tau\right) + \eta_{0j}\right] + \sum_{j=1}^{Q} \begin{cases} 0, \text{ if } \eta + w_j\tau \le \eta_{0j} - \dfrac{\pi}{2l_j} \\ c_j \cos^{\gamma_j+1}\left(l_j(\eta + w_j\tau - \eta_{0j})\right), \\ \text{if } \eta_{0j} - \dfrac{\pi}{2l_j} < \eta + w_j\tau \le \eta_{0j} + \dfrac{\pi}{2l_j}, \\ 0, \text{ if } \eta + w_j\tau > \eta_{0j} + \dfrac{\pi}{2l_j} \end{cases} \quad (58)$$

where a_0, B_i, K_i, β, C_j, L_j, λ, b_i, α_i, k_i, β_i, c_j, γ_j, l_j, w_j, x_{0i}, and η_{0j} are all arbitrary constants, and $M, N, P, Q, \alpha_i, \gamma_j$ are positive integers. Then the physical quantity Q (27) with (57) and (58) becomes a combined multi-dromion-compacton solution, which also shows a not-completely elastic interaction behavior. When choosing

$$\begin{aligned} &a_0 = 10,\ B_1 = 1/2,\ K_1 = 1,\ \beta = 1/2,\ C_1 = 1, \\ &L_1 = 1,\ \lambda = 0,\ b_1 = 1,\ \alpha_1 = 4,\ k_1 = 1, \beta_1 = -1, \\ &c_1 = 1,\ \gamma_1 = 4,\ l_1 = 1,\ w_1 = 0,\ x_{01} = 0,\ \eta_{01} = 0, \end{aligned} \quad (59)$$

we can derive a combined dromion-compacton localized coherent structure with not-completely elastic behavior depicted in Figure 5.

3.4. Completely Elastic Interactions

In fact, we can also derive some combined localized coherent structures with completely elastic interaction behavior by selecting u and φ appropriately. In this subsection, as an example, we consider some combined localized coherent structures of a peakon and a compacton solving the (2+1)-dimensional soliton system. One of the simplest choices may be

$$u = a_0 + \begin{cases} \sum_{i=1}^{M} d_i \exp(m_i x - \beta_i\tau + x_{0i}), \\ m_i x - \beta_i\tau + x_{0i} \le 0 \\ \sum_{i=1}^{M} \left(-d_i \exp(-m_i x + \beta_i\tau - x_{0i}) + 2d_i\right), \\ m_i x - \beta_i\tau + x_{0i} > 0, \end{cases} + \sum_{i=1}^{P} \begin{cases} 0, \text{ if } x + \lambda_i\tau \le x_{0i} - \dfrac{\pi}{2k_i} \\ b_i \cos^{a_i+1}\left(k_i(x + \lambda_i\tau - x_{0i})\right), \\ \text{if } x_{0i} - \dfrac{\pi}{2k_i} < x + \lambda_i\tau \le x_{0i} + \dfrac{\pi}{2k_i}, \\ 0, \text{ if } x + \lambda_i\tau > x_{0i} + \dfrac{\pi}{2k_i} \end{cases} \quad (60)$$

$$\varphi = \begin{cases} \sum_{j=1}^{N} e_j \exp(n_j\eta + w_j\tau + \eta_{0j}), \\ \text{if } n_j\eta + w_j\tau + \eta_{0j} \le 0 \\ \sum_{j=1}^{N} \left(-e_j \exp(-n_j\eta - w_j\tau - \eta_{0j}) + 2e_j\right), \\ \text{if } n_j\eta + w_j\tau + \eta_{0j} > 0, \end{cases} + \sum_{j=1}^{Q} \begin{cases} 0, \text{ if } \eta + \theta_j\tau \le \eta_{0j} - \dfrac{\pi}{2l_j} \\ c_j \cos^{\gamma_j+1}\left(l_j(\eta + \theta_j\tau - \eta_{0j})\right), \\ \text{if } \eta_{0j} - \dfrac{\pi}{2l_j} < \eta + \theta_j\tau \le \eta_{0j} + \dfrac{\pi}{2l_j}, \\ 0, \text{ if } \eta + \theta_j\tau > \eta_{0j} + \dfrac{\pi}{2l_j} \end{cases} \quad (61)$$

where a_0, d_i, m_i, β_i, e_j, n_j, w_j, b_i, α_i, k_i, λ_i, c_j, γ_j, l_j, θ_j, x_{0i}, and η_{0j} are all arbitrary constants, and $M, N, P, Q, \alpha_i, \gamma_j$ are positive integers. In this case the physical quantity Q (27) with (60) and (61) describes a combined multi-peakon-compacton solution with completely elastic behavior. When we fix the parameters as

$$\begin{aligned} &a_0 = 30,\ d_1 = 1,\ m_1 = 1,\ \beta_1 = -1,\ e_1 = 1, \\ &n_1 = 1,\ w_1 = 0,\ b_1 = 1,\ \alpha_1 = 4,\ k_1 = 1,\ \lambda_1 = -1, \\ &c_1 = 1,\ \gamma_1 = 4,\ l_1 = 1,\ \theta_1 = 0,\ x_{01} = 5,\ \eta_{01} = 5, \end{aligned} \quad (62)$$

the evolution of this combined multi-peakon-compacton solution with completely elastic behavior is displayed in Figure 6.

4. Summary

Starting from the known variable separated excitations which describe some quite universal (2+1)-dimensional physical model, of a (2+1)-dimensional soliton equation, we discuss the interactions among peakons, dromions, and compactons both analytically

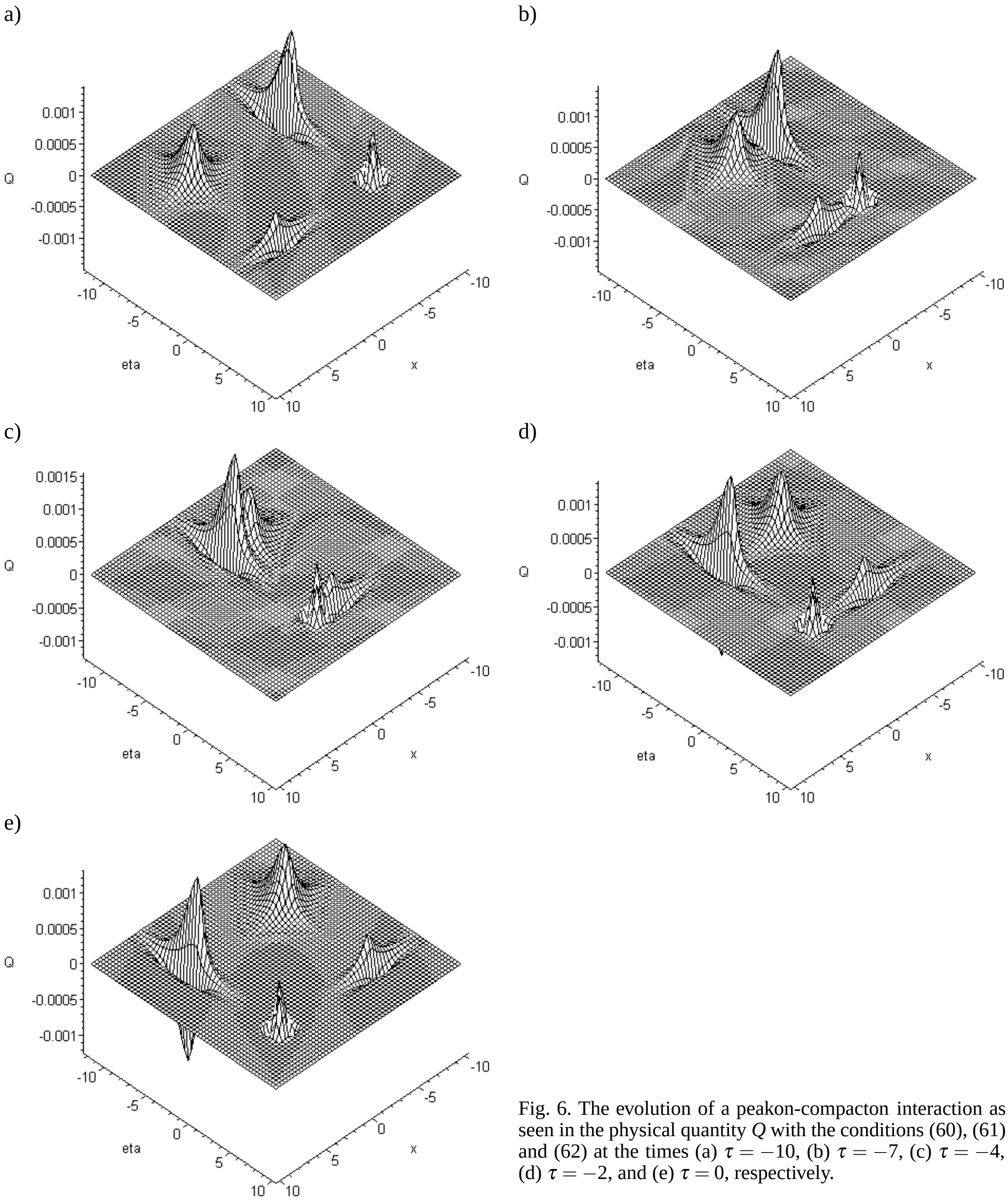


Fig. 6. The evolution of a peakon-compacton interaction as seen in the physical quantity Q with the conditions (60), (61) and (62) at the times (a) $\tau = -10$, (b) $\tau = -7$, (c) $\tau = -4$, (d) $\tau = -2$, and (e) $\tau = 0$, respectively.

and graphically, and reveal some novel properties and interesting behavior: the interactions for peakon-dromion, compacton-dromion, and peakon-compacton may be completely inelastic or not-completely elastic or completely elastic, depending on the specific details of the solutions. This paper is only an introductory attempt to describe these phenomena. Because of the wide applications of the soliton theory, to learn more about the interactions between different types of solitary waves and their applications in reality is worth further study.

Acknowledgement

The author would like to express his sincere thanks to the referees for their many helpful advices and suggestions. This work was supported by the National Natural Science Foundation of China and the Natural Science Foundation of Shandong Province in China.

[1] Eds. R.K. Bullough and P.J. Caudrey, Solitons, Springer-Verlag, Berlin 1980; R.K. Dodd, J.C. Eilbeck, and H.C. Morris, Solitons and Nonlinear Equations, Academic Press, London 1984; P.G. Drazin and R.S. Johnson, Solitons: An Introduction, Cambridge University Press, Cambridge, England 1988; C.H. Gu, Soliton Theory and Its Applications, Springer-Verlag, New York 1995; E. Infeld and G. Rowlands, Nonlinear Waves, Solitons, and Chaos, 2nd ed., Cambridge University Press, Cambridge, England 2000; C. Rogers and W.K. Schief, "Bäcklund and Darboux Transformations, Geometry and Modern Applications in Soliton Theory", Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge 2002.
[2] S. Y. Lou, Phys. Rev. Lett. **71**, 4099 (1993).
[3] A. Maccari, J. Math. Phys. **37**, 6207 (1996).
[4] K. Porsezian, J. Math. Phys. **38**, 4675 (1997).
[5] X. Y. Tang, C. L. Chen, and S. Y. Lou, J. Phys. A: Math. Gen. **35**, L293 (2002).
[6] S. Y. Lou, J. Phys. A: Math.Gen. **35**, 10619 (2002).
[7] C. L. Bai, J. Phys. Soc. Japan **73**, 37 (2004).
[8] C. L. Bai and H. Zhao, Z. Naturforsch. **59**a, 412 (2004).
[9] X. Y. Tang, S. Y. Lou, and Y. Zhang, Phys. Rev. E **66**, 46601 (2002).
[10] X. Y. Tang and S. Y. Lou, J. Math. Phys. **44**, 4000 (2003).
[11] S. Y. Lou, J. Phys. A: Math. Gen. **36**, 3877 (2003).